Badly approximable points on manifolds

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Abstract

Addressing a problem of Davenport we show that any finite intersection of the sets of weighted badly approximable points on any analytic nondegenerate manifold in $\mathbb{R}^n$ has full dimension. In particular, this settles the Schmidt conjecture on badly approximable points in arbitrary dimensions. As another application, the result also settles a problem of Bugeaud regarding real numbers badly approximable by algebraic numbers.

1 Introduction

Throughout this paper $\mathcal{R}_n$ denotes the set of $\mathbf{r}=(r_1,\dots,r_n)\in\mathbb{R}^n_{\ge0}$ such that $r_1+\ldots+r_n=1$. Given $\mathbf{r}\in\mathcal{R}_n$, the point $\mathbf{y}=(y_1,\dots,y_n)\in\mathbb{R}^n$ will be called $\mathbf{r}$-*badly approximable* if there exists $c>0$ such that for any $H\ge1$ the only integer solution $(a_0,a_1,\dots,a_n)$ to the system

$$|a_0+a_1y_1+\cdots+a_ny_n|<cH^{-1},\qquad |a_i|<H^{r_i}\quad(1\le i\le n)\tag{1}$$

is $a_0=\cdots=a_n=0$. The set of $\mathbf{r}$-badly approximable points in $\mathbb{R}^n$ will be denoted by $\mathbf{Bad}(\mathbf{r})$. When $r_1=\cdots=r_n=\frac{1}{n}$ we denote $\mathbf{Bad}(\mathbf{r})$ by $\mathbf{Bad}(n)$.

Using Mahler's version [Mah39] of Khintchine's transference (also see Appendix in [BPV11]), it is readily seen that $\mathbf{y}\in\mathbf{Bad}(\mathbf{r})$ if and only if there exists $c>0$ such that for any $Q\ge1$ the only integer solution $(q,p_1,\dots,p_n)$ to the system

$$|q|<Q,\qquad |qy_i-p_i|<\left(c\,Q^{-1}\right)^{r_i}\quad(1\le i\le n)\tag{2}$$

is $q=p_1=\cdots=p_n=0$. Both (1) and (2) can be rewritten in an asymptotic form. For instance, in relation to (2) we have that $\mathbf{y}\in\mathbf{Bad}(\mathbf{r})$ if and only if $\liminf\limits_{q\to\infty}\max\limits_{1\le i\le n}q\|qy_i\|^{1/r_i}>0$, where $\|x\|=\min\{|x-m|:m\in\mathbb{Z}\}$.

In 1964 Davenport [Dav64] established that for any C^1 map $\mathbf{f}:\mathbb{R}^m\to\mathbb{R}^n$ with $m\ge n$ and such that the Jacobian of $\mathbf{f}$ has rank n at some point, there exists continuum many $\mathbf{x}$ such that $\mathbf{f}(\mathbf{x})\in\mathbf{Bad}(n)$. Commenting on this result Davenport writes [Dav64, p. 52] "*Problems of a much more difficult character arise when the number of independent parameters is less than the dimension of simultaneous approximation. I do not know whether there is a set of α with the cardinal of the continuum such that the pair (α,α^2) is badly approximable for simultaneous approximation.*" Essentially, $m<n$ implies that $\mathbf{f}(\mathbf{x})$ lies on a submanifold of $\mathbb{R}^n$ and so Davenport raised the problem of investigating badly approximable points restricted to manifolds. Noteworthy, even the 'simplest'

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case of a parabola appeared to be a challenge. Addressing Davenport's problem will be the main content of this paper. More generally, we shall investigate countable intersections of $\mathbf{Bad}(\mathbf{r})$ restricted to manifolds.

Apart from Davenport's problem there are other strong motives for this study. For instance, in recent years there has been significant progress regarding $\mathbf{Bad}(\mathbf{r})$ restricted to the supports of the so-called absolutely friendly measures [KW05, KTV06, Fis09, KW10]. Nevertheless, the Riemannian measures on submanifolds in $\mathbb{R}^n$ do not fall into this category. Thus the progress for absolutely friendly measures makes Davenport's problem especially tempting. In another direction, Schmidt's conjecture that $\mathbf{Bad}(\frac{1}{3},\frac{2}{3})\cap\mathbf{Bad}(\frac{2}{3},\frac{1}{3})\neq\emptyset$ has led to a surge of activity in studying the intersections of the sets $\mathbf{Bad}(\mathbf{r})$ – see [PV02, KTV06, KW10]. The conjecture was open for 30 years but the recent progress has been dramatic. First, Badziahin, Pollington and Velani [BPV11] have shown that for any sequence $\mathbf{r}_k=(i_k,j_k)\in\mathcal{R}_2$ such that

$$\liminf_{k\to\infty}\min\{i_k,j_k\}>0 \tag{3}$$

and any vertical line $L_\theta=\{(\theta,y):y\in\mathbb{R}\}\subset\mathbb{R}^2$ with $\theta\in\mathbf{Bad}(1)$ one has that

$$\dim\left(\bigcap_k\mathbf{Bad}(\mathbf{r}_k)\cap L_\theta\right)=1. \tag{4}$$

This is already much stronger than Schmidt's original conjecture. Yet more recently An [An13] has removed condition (3) from the above statement by establishing the stronger property that the set $\mathbf{Bad}(i,j)\cap L_\theta$, where $i+j=1$, $i,j\geq 0$, is wining for a so-called Schmidt game in L_θ. Amazingly, in a related paper An [An] proves that $\mathbf{Bad}(i,j)$ is winning for a 2-dimensional Schmidt game, thus providing one more proof of Schmidt's conjecture. Generalising [BPV11] in yet another direction Nesharim [Nes] has proved that the set in the left hand side of (4) intersected with naturally occurring fractals embedded in L_θ is uncountable for any sequence $(\mathbf{r}_k)_{k\in\mathbb{N}}$. Finally, continuing the spectacular success of [BPV11] Badziahin and Velani [BV] have proved (4) with L_θ replaced by a fairly generic curve or line in $\mathbb{R}^2$ provided that (3) is valid. This settles Davenport's problem in dimension $n=2$, and in particular answers his specific question regarding badly approximable points (α,α^2) on the parabola. However, until now no progress has been made for $n>2$.

In this paper we resolve Davenport's problem for arbitrary analytic nondegenerate manifolds and as a byproduct prove Schmidt's conjecture in arbitrary dimensions. The key new ideas introduced in this paper lie within counting techniques for constructing Cantor type sets of badly approximable points. In fact, even for $n=2$ the proofs given are of independent interest.

2 Main results and corollaries

The results will apply to nondegenerate maps and manifolds in $\mathbb{R}^n$, that is the manifolds that are immersed into $\mathbb{R}^n$ by nondegenerate maps. Formally, an analytic map $\mathbf{f}:B\to\mathbb{R}^n$ defined on a ball $B\subset\mathbb{R}^m$ will be called *nondegenerate* if the functions $1,f_1,\dots,f_n$ are linearly independent over $\mathbb{R}$. Thus, a connected analytic manifold $\mathcal{M}\subset\mathbb{R}^n$ is nondegenerate if it is not contained in a proper affine subspace of $\mathbb{R}^n$.

Throughout the paper $n \geq 2$ is an integer and $\mathcal{F}_n(B)$ denotes a finite family of maps $\mathbf{f} : B \to \mathbb{R}^n$ with the common domain B. To avoid ambiguity, let us agree from the beginning that all intervals and balls mentioned in this paper are of positive and finite diameter. Given $\mathbf{r} = (r_1, \ldots, r_n) \in \mathcal{R}_n$, let

$$\tau(\mathbf{r}) \stackrel{\text{def}}{=} \min\{r_i : r_i \neq 0\}. \tag{5}$$

Theorem 1 *Let B be an open ball in $\mathbb{R}^m$ with $1 \leq m \leq n$ and let $\mathcal{F}_n(B)$ be a finite family of analytic nondegenerate maps. For $k \in \mathbb{N}$ let $\mathbf{r}_k \in \mathcal{R}_n$ and*

$$\inf_k \tau(\mathbf{r}_k) > 0\,. \tag{6}$$

Then there exists a set $S \subset B$ such that $\dim S = m$ and

$$\mathbf{f}(S) \subset \mathbf{Bad}(\mathbf{r}_k) \qquad \text{for all } k \in \mathbb{N} \text{ and all } \mathbf{f} \in \mathcal{F}_n(B)\,. \tag{7}$$

Consequently, if $\mathcal{M}$ is an analytic nondegenerate manifold in $\mathbb{R}^n$, then

$$\dim \textstyle\bigcap_{k \in \mathbb{N}} \mathbf{Bad}(\mathbf{r}_k) \cap \mathcal{M} = \dim \mathcal{M}\,. \tag{8}$$

Condition (6) matches (3) and is satisfied for any finite sequence $\mathbf{r}_k$. Taking $\mathcal{M} = \mathbb{R}^n$ extends Schmidt's conjecture to arbitrary dimensions:

Corollary 1 *Assuming* (6) *we have that* $\dim \bigcap_{k \in \mathbb{N}} \mathbf{Bad}(\mathbf{r}_k) = n$.

Taking a single $\mathbf{f} = (x, x^2, \ldots, x^n)$ yields another corollary of high interest:

Corollary 2 *Assuming* (6), *for any interval $I \subset \mathbb{R}$ there is a set $S_I \subset I$ with $\dim S_I = 1$ and such that $(x, x^2, \ldots, x^n) \in \mathbf{Bad}(\mathbf{r}_k)$ for all $x \in S_I$ and $k \in \mathbb{N}$.*

Taking $\mathbf{r} = (\frac{1}{n}, \ldots, \frac{1}{n})$ in Corollary 2 settles Problems 24, 25 and 26 in [Bug04] on the existence of transcendental real numbers badly approximable by algebraic numbers of degree $\leq n$ (we skip the details as they are standard and will be considered in a subsequent paper along with some generalisations). Moreover, using Corollary 2 with $\mathbf{r}_k = (\frac{1}{k}, \ldots, \frac{1}{k}, 0, \ldots, 0) \in \mathcal{R}_n$ $(k = 1, \ldots, n)$, where the number of zeros is $n - k$, shows that the set of real numbers simultaneously badly approximable by algebraic numbers of degree $k \leq n$ is non-empty and indeed of maximal dimension:

Corollary 3 *Let $H(\alpha)$ denote the height of α (the maximal absolute value of the coefficients of the minimal polynomial of α over $\mathbb{Z}$). Then for any $n \geq 1$ and any interval $I \subset \mathbb{R}$ one has that*

$$\dim \bigcap_{k=1}^{n} \left\{ x \in I : \begin{array}{l} \exists\, c > 0 \quad |x - \alpha| \geq cH(\alpha)^{-k-1} \\ \textit{for all algebraic } \alpha \in \mathbb{C},\ \deg \alpha \leq k \end{array} \right\} = 1. \tag{9}$$

The sets in (9) are known to have Lebesgue measure zero, e.g., by a Khintchine type theorem proved in [Ber99]. Recall that the case $n = 1$ is a famous theorem of Jarník [Jar28] and the case $n = 2$ is proved in [BV]. An appealing problem is to show that (9) holds with the intersection taken over k from 1 to ∞.

When $m = 1$ the nondegeneracy of an analytic map $\mathbf{f} = (f_1, \dots, f_n)$ is equivalent to the Wronskian of $f_1', \dots, f_n'$ being not identically zero. More generally, the map $\mathbf{f}$ (not necessarily analytic) defined on an interval $I \subset \mathbb{R}$ will be called *nondegenerate at* $x_0 \in I$ if $\mathbf{f}$ is C^n on a neighborhood of x_0 and the Wronskian of $f_1', \dots, f_n'$ does not vanish at x_0. This definition of nondegeneracy at a single point is adopted within the following more general result for curves.

Theorem 2 *Let $\mathcal{F}_n(I)$ be a finite family of maps defined on an open interval $I \subset \mathbb{R}$ and nondegenerate at the same point $x_0 \in I$. For $k \in \mathbb{N}$ let $\mathbf{r}_k \in \mathcal{R}_n$ and let* (6) *be satisfied. Then there is a subset $S \subset I$ with $\dim S = 1$ such that*

$$\mathbf{f}(S) \subset \mathbf{Bad}(\mathbf{r}_k) \qquad \text{for all } k \in \mathbb{N} \text{ and all } \mathbf{f} \in \mathcal{F}_n(I)\,. \tag{10}$$

Theorem 2 implies Theorem 1. This is a simple application of the fibering technique given in [Spr80, pp. 9-10], which we briefly repeat here with the necessary modifications. Let $\mathcal{F}_n(B)$ be as in Theorem 1 and let $\mathbf{f} \in \mathcal{F}_n(B)$. Assume that $m \geq 2$ as otherwise there is nothing to show. The change of variables

$$(x_1, \dots, x_m) = (t, u_2 t^d, u_3 t^{d^2}, \dots, u_m t^{d^{m-1}})$$

takes B into some open domain in $\mathbb{R}^m$. Let D be a ball in this domain. For each $\mathbf{u} = (u_2, \dots, u_m) \in \mathbb{R}^{m-1}$ let $I_\mathbf{u} = \{t \in \mathbb{R} : (t, u_2, \dots, u_m) \in D\}$. Let $D' = \{\mathbf{u} \in \mathbb{R}^{m-1} : I_\mathbf{u} \neq \emptyset\}$. It is easily seen that D' is a non-empty open ball in $\mathbb{R}^{m-1}$ and $I_\mathbf{u}$ is a non-empty open interval for each $\mathbf{u} \in D'$.

Since $\mathbf{f}$ is analytic and nondegenerate, for all sufficiently large $d \in \mathbb{N}$ and every $\mathbf{u} = (u_2, \dots, u_m) \in D'$ the map $\mathbf{f}_\mathbf{u}(t) = \mathbf{f}(t, u_2 t^d, u_3 t^{d^2}, \dots, u_m t^{d^{m-1}})$ defined on $I_\mathbf{u}$ is analytic and nondegenerate – see [Spr80, pp. 9-10]. Since $\mathcal{F}_n(B)$ is finite, we can choose d sufficiently large so that the above claim is true for any choice of $\mathbf{f} \in \mathcal{F}_n(B)$. Hence, for each $\mathbf{u} \in D'$ Theorem 2 is applicable with $\mathcal{F}_n(I_\mathbf{u}) = \{\mathbf{f}_\mathbf{u} : \mathbf{f} \in \mathcal{F}_n(B)\}$. It follows that for each $\mathbf{u} \in D'$ there exists $S_\mathbf{u} \subset I_\mathbf{u}$ with $\dim S_\mathbf{u} = 1$ and $\mathbf{f}_\mathbf{u}(S_\mathbf{u}) \subset \mathbf{Bad}(\mathbf{r}_k)$ for all $k \in \mathbb{N}$ and $\mathbf{f}_\mathbf{u} \in \mathcal{F}_n(I_\mathbf{u})$. Then the set

$$S = \big\{(u_1, \mathbf{u}) \in \mathbb{R}^m : \mathbf{u} \in D',\ u_1 \in S_\mathbf{u}\big\}$$

satisfies $\mathbf{f}(S) \subset \mathbf{Bad}(\mathbf{r}_k)$ for all $k \in \mathbb{N}$ and $\mathbf{f} \in \mathcal{F}_n(B)$. By Marstrand's slicing lemma (see Corollary 7.12 in [Fal03]), we have that $\dim S \geq \dim D' + 1 = m$, while trivially $\dim S \leq m$. Thus, $\dim S = m$ and the implication is shown. $\boxtimes$

Remarks. Before we fully plunge into the proof of Theorem 2, we discuss possible generalisations and further problems. First of all, the analyticity assumption within Theorem 1 can be relaxed by making use of more general fibering techniques such as that of [Pya69]. For now, we opt to leave this purely technical task to the interested reader. Thus, modulo condition (6) the results of this paper settle Conjectures A and C from [BV, §1.3] for nondegenerate manifolds that are sufficiently smooth. Yet the techniques developed in this paper open up a number of exciting research avenues. For example, beyond nondegenerate manifolds, it will be tempting to obtain generalisations of Theorems 1 and 2 for friendly measures as defined in [KLW04] as well as for affine subspaces of $\mathbb{R}^n$ and their submanifolds – see [Kle03] for a related context. In another direction, it would be interesting to develop the theory of badly approximable

systems of linear forms. Removing condition (6) is another appealing problem that would be settled if the sets of interest were shown to be winning in the sense of Schmidt (see [Sch80], [BV, §1.3] and [An13]). However, the techniques of this paper could also help accomplishing this task: the key is to make the lower bound on M appearing in Theorem 7 below independent of $\tau(\mathbf{r})$. Finally, all of the above questions make sense and are of course interesting in the case of Diophantine approximation over $\mathbb{Q}_p$ and in positive characteristic.

3 Counting lattice points

The rest of the paper will be concerned with the proof of Theorem 2, which will rely heavily on efficient counting of lattice points in convex bodies. The lattices will arise upon reformulating $\mathbf{Bad}(\mathbf{r})$ in the spirit of Dani [Dan85] and Kleinbock [Kle98]. This will require the following notation. Given a subset Λ of $\mathbb{R}^{n+1}$, let

$$\delta(\Lambda) = \inf_{\mathbf{a}\in\Lambda\setminus\{\mathbf{0}\}} \|\mathbf{a}\|_\infty\,, \tag{11}$$

where $\|\mathbf{a}\|_\infty = \max\{|a_0|,\dots,|a_n|\}$ for $\mathbf{a} = (a_0,\dots,a_n)$. Given $0<\kappa<1$, let

$$G(\kappa;\mathbf{y}) = \begin{pmatrix} \kappa^{-1} & \kappa^{-1}\mathbf{y} \\ 0 & I_n \end{pmatrix}, \tag{12}$$

where $\mathbf{y}\in\mathbb{R}^n$ is regarded as a row and I_n is the $n\times n$ identity matrix. Finally, given $\mathbf{r}\in\mathcal{R}_n$, $b>1$ and $t\in\mathbb{N}$, define the $(n+1)\times(n+1)$ unimodular diagonal matrix

$$g^t_{\mathbf{r},b} = \mathrm{diag}\{b^t, b^{-r_1 t},\dots,b^{-r_n t}\}. \tag{13}$$

Lemma 1 *Let $\mathbf{y}\in\mathbb{R}^n$, $\mathbf{r}\in\mathcal{R}_n$. Then $\mathbf{y}\in\mathbf{Bad}(\mathbf{r})$ if and only if there exists $\kappa\in(0,1)$ and $b>1$ such that for all $t\in\mathbb{N}$*

$$\delta(g^t_{\mathbf{r},b}G(\kappa;\mathbf{y})\mathbb{Z}^{n+1}) \ge 1. \tag{14}$$

Proof. The necessity is straightforward as all one has to do is to take $H=b^t$ and divide each inequality in (1) by its right hand side. Then, assuming that $\mathbf{y}\in\mathbf{Bad}(\mathbf{r})$, the non-existence of integer solutions to (1) would imply (14) with $\kappa=c$. The sufficiency is only slightly harder. Assume that for some κ and b inequality (14) holds for all $t\in\mathbb{N}$, while $\mathbf{y}\notin\mathbf{Bad}(\mathbf{r})$. Take $c=\kappa/b$. By definition, there is an $H>1$ such that (1) has a non-zero integer solution $(a_0,\dots,a_n)$. Take $t=[\log H/\log b]+1$, where $[\,\cdot\,]$ denotes the integer part. Note that $Hb^{-t}<1$ and $H^{-1}b^t\le b$. Then (1) implies that $\delta(g^t_{\mathbf{r},b}G(\kappa;\mathbf{y})\mathbb{Z}^{n+1})<1$, contrary to (14). The proof is thus complete. ⊠

We proceed by recalling two classical results from the geometry of numbers. In what follows, $\mathrm{vol}_\ell(X)$ denotes the ℓ-dimensional volume of $X\subset\mathbb{R}^\ell$ and $\#X$ denotes the cardinality of X. Also $\det\Lambda$ will denote the *determinant* or *covolume* of a lattice Λ.

Minkowski's Convex Body Theorem (see [Sch80, Theorem 2B]) *Let $K\subset\mathbb{R}^\ell$ be a convex body symmetric about the origin and let Λ be a lattice in $\mathbb{R}^\ell$. Suppose that $\mathrm{vol}_\ell(K)>2^\ell\det\Lambda$. Then K contains a non-zero point of Λ.*

Theorem (Blichfeldt [Bli21]) *Let $K \subset \mathbb{R}^\ell$ be a convex bounded body and let Λ be a lattice in $\mathbb{R}^\ell$ such that* $\operatorname{rank}(K \cap \Lambda) = \ell$. *Then*

$$\#(K \cap \Lambda) \;\le\; \ell!\,\frac{\operatorname{vol}_\ell(K)}{\det \Lambda} + \ell.$$

The following lemma is a straightforward consequence of Blichfeldt's theorem.

Lemma 2 (cf. Lemma 4 in [KTV06]) *Let K be a bounded convex body in $\mathbb{R}^\ell$ with $0 \in K$ and $\operatorname{vol}_\ell(K) < 1/\ell!$. Then* $\operatorname{rank}(K \cap \mathbb{Z}^\ell) \le \ell - 1$.

The bodies K of interest will arise as the intersection of parallelepipeds

$$\Pi_{\boldsymbol{\theta}} = \big\{\mathbf{x} = (x_0, \dots, x_n) \in \mathbb{R}^{n+1} : |x_i| < \theta_i, \quad i = 0, \dots, n\big\} \tag{15}$$

with ℓ-dimensional subspaces of $\mathbb{R}^{n+1}$, where $\boldsymbol{\theta} = (\theta_0, \dots, \theta_n)$ is an $(n+1)$-tuple of positive numbers. In view of this, we now obtain an estimate for the volume of the bodies that arise this way (Lemma 3 below) and then verify what Blichfeldt's theorem means for such bodies (Lemma 4 below).

Lemma 3 *Let $\ell \in \mathbb{N}$, $\ell \le n+1$, $\boldsymbol{\theta} = (\theta_0, \dots, \theta_n)$ with $\theta_0, \dots, \theta_n > 0$. Then for any linear subspace V of $\mathbb{R}^{n+1}$ of dimension ℓ we have that*

$$\operatorname{vol}_\ell(\Pi_{\boldsymbol{\theta}} \cap V) \le 2^\ell (n+1)^{\ell/2} \Theta_\ell, \qquad \textit{where} \qquad \Theta_\ell = \max_{\substack{I \subset \{0,\dots,n\} \\ \#I = \ell}} \prod_{i \in I} \theta_i.$$

Proof. Since V is a linear subspace of $\mathbb{R}^{n+1}$ of dimension ℓ, it is given by $n+1-\ell$ linear equations. Using Gaussian elimination, we can rewrite these equations to parametrise V with a linear map $\mathbf{f} : \mathbb{R}^\ell \to \mathbb{R}^{n+1}$ of $x_{i_1}, \dots, x_{i_\ell}$ such that

$$\mathbf{f}(x_{i_1}, \dots, x_{i_\ell}) = (x_{i_1}, \dots, x_{i_\ell}) M,$$

where $M = (m_{i,j})$ is an $\ell \times (n+1)$ matrix with $|m_{i,j}| \le 1$ for all i and j. Then note that $\operatorname{vol}_\ell(\Pi_{\boldsymbol{\theta}} \cap V)$ is bounded by the area of the intersection of V with the cylinder $|x_{i_j}| \le \theta_{i_j}$ for $j = 1, \dots, \ell$. This area is equal to

$$\int_{-\theta_{i_1}}^{\theta_{i_1}} \cdots \int_{-\theta_{i_\ell}}^{\theta_{i_\ell}} \left\| \frac{\partial \mathbf{f}}{\partial x_{i_1}} \wedge \dots \wedge \frac{\partial \mathbf{f}}{\partial x_{i_\ell}} \right\|_e dx_{i_1} \dots dx_{i_\ell}\,, \tag{16}$$

where $\|\cdot\|_e$ is the Euclidean norm on $\bigwedge^\ell(\mathbb{R}^{n+1})$. Since $|m_{i,j}| \le 1$, every coordinate of every partial derivative of $\mathbf{f}$ is bounded by 1 in absolute value. Hence $\|\partial \mathbf{f}/\partial x_{i_j}\|_e \le \sqrt{n+1}$ and the integrand in (16) is bounded above by $(\sqrt{n+1})^\ell$. This readily implies that the area given by (16) is bounded above by $2^\ell (n+1)^{\ell/2} \theta_{i_1} \cdots \theta_{i_\ell} \le 2^\ell (n+1)^{\ell/2} \Theta_\ell$, whence the result follows. ⊠

Lemma 4 *Let $c(n) = 4^{n+1}(n+1)^{(n+1)/2}(n+1)!$ and let $\boldsymbol{\theta}$ and Θ_ℓ be as in Lemma 3. Then for any discrete subgroup Γ of $\mathbb{R}^{n+1}$ with $\ell = \operatorname{rank}\big(\Gamma \cap \Pi_{\boldsymbol{\theta}}\big) > 0$ we have that*

$$\#\big(\Gamma \cap \Pi_{\boldsymbol{\theta}}\big) \;\le\; c(n)\,\frac{\Theta_\ell}{\delta(\Gamma)^\ell} + n + 1. \tag{17}$$

Proof. Let $V = \mathrm{span}(\Gamma \cap \Pi_{\boldsymbol{\theta}})$ and $\Lambda = V \cap \Gamma$. Clearly, $\mathrm{rank}\,(\Lambda) = \ell$ and furthermore Λ is a lattice in V. Also note that $\Gamma \cap \Pi_{\boldsymbol{\theta}} = \Lambda \cap \Pi_{\boldsymbol{\theta}}$. Since $\Lambda \subseteq \Gamma$, we have that $\delta(\Gamma) \leq \delta(\Lambda)$. Let $B(r)$ denote the open ball in V of radius r centred at the origin. Note that the length of any non-zero point in Λ is bigger than or equal to $\delta(\Lambda) \geq \delta(\Gamma)$. Hence, by Minkowski's convex bodies theorem, we must have that $\mathrm{vol}_\ell\big(B(\delta(\Gamma)\big) \leq 2^\ell \det \Lambda$, whence we obtain $\det \Lambda \geq \mathrm{vol}_\ell(B(\delta(\Lambda)))2^{-\ell} \geq (\delta(\Lambda)/2)^\ell$. Now using this inequality, Blichfeldt's theorem, Lemma 3 and the fact that $\ell \leq n+1$ readily gives (17). ⊠

We are now approaching the key counting result of this section. Let

$$\Pi(b,u) \stackrel{\mathrm{def}}{=} \Pi_{\boldsymbol{\theta}} \qquad \text{with} \quad \boldsymbol{\theta} = (b^u, 1, \dots, 1), \tag{18}$$

where $u > 0$, $b > 1$ and $\Pi_{\boldsymbol{\theta}}$ is given by (15). Given $\mathbf{r} \in \mathcal{R}_n$, let

$$z(\mathbf{r}) \stackrel{\mathrm{def}}{=} \#\{\, i \,:\, r_i = 0 \,\} \qquad \text{and} \qquad \lambda(\mathbf{r}) \stackrel{\mathrm{def}}{=} \big(1 + \tau(\mathbf{r})\big)^{-1}. \tag{19}$$

Recall that $\tau(\mathbf{r})$, $\delta(\cdot)$, $g^t_{\mathbf{r},b}$ and $\Pi(b,u)$ are given by (5), (11), (13) and (18) respectively, and $[x]$ denotes the integer part of x.

Lemma 5 *Let $b > 1$, $\mathbf{r} \in \mathcal{R}_n$, $\lambda = \lambda(\mathbf{r})$, $z = z(\mathbf{r})$, $t \in \mathbb{N}$, $u \in \mathbb{R}$, $1 \leq \lambda u \leq t$ and $c(n)$ be as in Lemma 4. Let $g^t = g^t_{\mathbf{r},b}$. Let Λ be a discrete subgroup of $\mathbb{R}^{n+1}$ such that* $\mathrm{rank}\,\Lambda \leq n - z$ *and*

$$\delta\big(g^{t-[\lambda u]}\Lambda\big) \geq 1. \tag{20}$$

Then

$$\#(g^t\Lambda) \cap \Pi(b,u) \leq 2c(n) b^\tau b^{\lambda u}. \tag{21}$$

Proof. Let $\mathbf{x} = (x_0, \dots, x_n) \in \Lambda$ be such that $g^t\mathbf{x} \in \Pi(b,u)$. By the definitions of $g^t = g^t_{\mathbf{r},b}$ and $\Pi(b,u)$, we have that $b^t|x_0| < b^u$ and $b^{-r_i t}|x_i| < 1$ for $i = 1, \dots, n$. Equivalently, for $s \in \mathbb{Z}$, $1 \leq s \leq u-1$, we have that

$$b^{t-s}|x_0| < b^{u-s} \qquad \text{and} \qquad b^{-r_i(t-s)}|x_i| < b^{r_i s} \quad (1 \leq i \leq n).$$

This can be writhen as $g^{t-s}\mathbf{x} \in \Pi_{\boldsymbol{\theta}}$, where $\boldsymbol{\theta} = (b^{u-s}, b^{r_1 s}, \dots, b^{r_n s})$. Therefore,

$$(g^t\Lambda) \cap \Pi(b,u) \;=\; \Gamma \cap \Pi_{\boldsymbol{\theta}}, \tag{22}$$

where $\Gamma = g^{t-s}\Lambda$. Now take $s = [\lambda u]$. Then, by the left hand side of (20), we have that $\delta(\Gamma) \geq 1$. Hence, by Lemma 4 and (22), we get

$$\#(g^t\Lambda) \cap \Pi(b,u) \;=\; \#\big(\Gamma \cap \Pi_{\boldsymbol{\theta}}\big) \;\leq\; c(n)\Theta_\ell + n + 1, \tag{23}$$

where $\ell = \mathrm{rank}\,\Gamma = \mathrm{rank}\,\Lambda \leq n - z$. Note that all the components of $\boldsymbol{\theta}$ are ≥ 1 and exactly z of them equal 1. Then, since $\ell \leq n - z$ and $s = [\lambda u]$, we get that

$$\Theta_\ell \;\leq\; \frac{\theta_0 \dots \theta_n}{\min\{\theta_i > 1\}} = \frac{b^u}{\min\{b^{u-s}, b^{\tau s}\}} \leq \max\{b^{\lambda u}, b^{u-\tau(\lambda u - 1)}\} = b^\tau b^{\lambda u}.$$

Combining this estimate with (23) and the obvious fact that $n + 1 < c(n) b^\tau b^{\lambda u}$ gives (21). ⊠

4 'Dangerous' intervals

In view of Lemma 1, when constructing the set S for Theorem 2 we will aim to avoid the solutions of the inequalities $\delta(g^t_{\mathbf{r},b}G_x\mathbb{Z}^{n+1}) < 1$, where $G_x = G(\kappa;\mathbf{y})$ with $\mathbf{y} = \mathbf{f}(x)$ and κ is a sufficiently small constant. For fixed $\mathbf{r}, b, t, \mathbf{f}$ and κ the above inequality is equivalent to the existence of $(a_0, \mathbf{a}) \in \mathbb{Z}^{n+1}$ with $\mathbf{a} \neq \mathbf{0}$ satisfying

$$\left\{\begin{array}{rcl} |a_0 + \mathbf{a}.\mathbf{f}(x)| & < & \kappa b^{-t}, \\ |a_i| & < & b^{r_i t} \;\; (1 \le i \le n). \end{array}\right. \tag{24}$$

Here the dot means the usual inner product. That is $\mathbf{a}.\mathbf{b} = a_1b_1 + \cdots + a_nb_n$ for any given $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$. In this section we study intervals arising from (24) that, for obvious reasons, are referred to as *dangerous* (see [Sch80] for similar terminology). We will consider several cases that are tied up with the magnitude of $\mathbf{a}.\mathbf{f}'(x)$; *e.i.*, the derivative of $a_0 + \mathbf{a}.\mathbf{f}(x)$, – see Propositions 1 and 2 below.

Throughout $\mathcal{F}_n(I)$ and x_0 are as in Theorem 2. First we discuss some conditions that arise from the nondegeneracy assumption on maps in $\mathcal{F}_n(I)$. Let $\mathbf{f} = (f_1, \ldots, f_n) \in \mathcal{F}_n(I)$. Since $\mathbf{f}$ is nondegenerate at $x_0 \in I$, there is a sufficiently small neighborhood $I_\mathbf{f}$ of x_0 such that the Wronskian of $f'_1, \ldots, f'_n$, which, by definition, is the determinant $\det\big(f_j^{(i)}\big)_{1\le i,j\le n}$, is non zero everywhere in $I_\mathbf{f}$. Then every coordinate function f_j is non-vanishing at all but countably many points of $I_\mathbf{f} \subset I$ – see, e.g., [BB96, Lemma 3]. Since $\mathbf{f} \in C^n$ and $\mathcal{F}_n(I)$ is finite, we can choose a compact interval $I_0 \subset \bigcap_{\mathbf{f}\in\mathcal{F}_n(I)} I_\mathbf{f} \subset I$ satisfying

Property F: There are constants $0 < c_0 < 1 < c_1$ such that for every map $\mathbf{f} = (f_1, \ldots, f_n) \in \mathcal{F}_n(I)$, for all $x \in I_0$, $1 \le i \le n$ and $0 \le j \le n$ one has that

$$\left|\det\big(f_j^{(i)}(x)\big)_{1\le i,j\le n}\right| > c_0, \qquad |f'_j(x)| > c_0 \qquad \text{and} \qquad |f_j^{(i)}(x)| < c_1. \tag{25}$$

Next, we prove two auxiliary lemmas that are well known in a related context.

Lemma 6 (cf. Lemma 5 in [BB96]) *Let $I_0 \subset I$ be a compact interval satisfying Property F. Let $2c_2 = c_0c_1^{-n+1}n!^{-1}$, where c_0 and c_1 arise from* (25). *Then for any $\mathbf{f} \in \mathcal{F}_n(I)$, any $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n \setminus \{0\}$ and any $x \in I_0$ there exists $i \in \{1, \ldots, n\}$ such that $|\mathbf{a}.\mathbf{f}^{(i)}(x)| \ge 2c_2 \max_{1\le j\le n} |a_j|$.*

Proof. Solving the system $a_1f_1^{(i)}(x)+\cdots+a_nf_n^{(i)}(x) = \mathbf{a}.\mathbf{f}^{(i)}(x)$, where $1 \le i \le n$, by Cramer's rule with respect to a_i and using (25) to estimate the determinants involved in the rule we obtain

$$|a_j| \;\le\; c_1^{n-1} \cdot n!c_0^{-1} \max_{1\le i\le n} |\mathbf{a}.\mathbf{f}^{(i)}(x)|$$

for each $j = 1, \ldots, n$, whence the statement of lemma readily follows. ⊠

Lemma 7 (cf. Lemma 6 in [BB96]) *Let $I_0 \subset I$ and c_2 be as in Lemma 6. Then there is $\delta_0 > 0$ such that for any interval $J \subset I_0$ of length $|J| \le \delta_0$, any $\mathbf{f} \in \mathcal{F}_n(I)$ and $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n \setminus \{0\}$, there is an $i \in \{1, \ldots, n\}$ satisfying*

$$\inf_{x\in J} |\mathbf{a}.\mathbf{f}^{(i)}(x)| \ge c_2 \max_{1\le j\le n} |a_j|. \tag{26}$$

Proof. Since I_0 is compact, for each $\mathbf{f} \in \mathcal{F}_n(I)$ and $1 \le i \le n$, the map $\mathbf{f}^{(i)}$ is uniformly continuous on I_0. Hence, there is a $\delta_{i,\mathbf{f}} > 0$ such that for any $x, y \in I_0$ with $|x - y| \le \delta_{i,\mathbf{f}}$ we have $|\mathbf{f}^{(i)}(x) - \mathbf{f}^{(i)}(y)| < c_2/n$. Let $J \subset I_0$ be an interval of length $|J| \le \delta_{i,\mathbf{f}}$ and $x, y \in J$. By Lemma 6, there is $i \in \{1, \dots, n\}$ such that $|\mathbf{a}.\mathbf{f}^{(i)}(x)| \ \ge\ 2c_2 h$, where $h = \max_{1\le j\le n} |a_j|$. Then

$$|\mathbf{a}.\mathbf{f}^{(i)}(y)| \ge |\mathbf{a}.\mathbf{f}^{(i)}(x)| - |\mathbf{a}.\mathbf{f}^{(i)}(y) - \mathbf{a}.\mathbf{f}^{(i)}(x)| \ge 2c_2h - nhc_2/n = c_2h. \tag{27}$$

Since $\mathcal{F}_n(I)$ is finite, $\delta_0 = \inf_{i,\mathbf{f}} \delta_{i,\mathbf{f}} > 0$. Hence (27) implies (26) provided that $|J| \le \delta_0$.

$\boxtimes$

Proposition 1 *Let $I_0 \subset I$ be a compact interval satisfying Property F and $\mathbf{f} \in \mathcal{F}_n(I)$. Further, let δ_0 be as in Lemma 7, $\mathbf{r} \in \mathcal{R}_n$ and*

$$\gamma = \gamma(\mathbf{r}) \stackrel{\text{def}}{=} \max\{r_1, \dots, r_n\}. \tag{28}$$

Finally, let $t \in \mathbb{N}$, $\ell \in \mathbb{Z}_{\ge 0}$, $b > 1$, $\mathbf{a} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$, $a_0 \in \mathbb{Z}$, $0 < \kappa < 1$ and

$$D^1_{t,\ell,\mathbf{r},b,\kappa,\mathbf{f}}(a_0, \mathbf{a}) = \left\{ x \in I_0 \ :\ \begin{array}{rcl} |a_0 + \mathbf{a}.\mathbf{f}(x)| & < & \kappa b^{-t} \\ b^{\gamma t - (1+\gamma)\ell} \le\ |\mathbf{a}.\mathbf{f}'(x)| & < & b^{\gamma t - (1+\gamma)(\ell - 1)} \\ |a_i| & < & b^{r_i t} \end{array} \right\}.$$

Then, there is a constant $c_3 > 0$ depending on n, $|I_0|$, c_1, c_2 and δ_0 only such that the set $D^1_{t,\ell,\mathbf{r},b,\kappa,\mathbf{f}}(a_0, \mathbf{a})$ can be covered by a collection $\mathcal{D}^1_{t,\ell,\mathbf{r},b,\kappa,\mathbf{f}}(a_0, \mathbf{a})$ of at most c_3 intervals Δ of length $|\Delta| \le \kappa b^{-(1+\gamma)(t-\ell)}$.

Proof. We will abbreviate $D^1_{t,\ell,\mathbf{r},b,\kappa,\mathbf{f}}(a_0, \mathbf{a})$ as D^1 and naturally assume that $D^1 \ne \emptyset$ as otherwise there is nothing to prove. Since I_0 can be covered by at most $[\delta_0^{-1}|I_0|] + 1$ intervals J of length $|J| \le \delta_0$, it suffices to prove the proposition under the assumption that $|I_0| \le \delta_0$. Let $f(x) = a_0 + \mathbf{a}.\mathbf{f}(x)$. Then, by Lemma 7, we have that $|f^{(i)}(x)| > 0$ for a fixed $i \in \{1, \dots, n\}$ and all $x \in I_0$. First consider the case $i > 1$. Then, using Rolle's theorem, one finds that the function $f^{(j)}(x)$ vanishes on I_0 at $\le i - j$ points $(0 \le j \le i - 1)$. Assuming that $I_0 = [a, b]$, let $x_0 = a < x_1 < \cdots < x_{s-1} < x_s = b$ be the collection consisting of the points a and b and all the zeros of $\prod_{j=0}^{i-1} f^{(j)}(x)$. Then, as we have just seen $s \le 1 + \sum_{j=0}^{i-1}(i - j) = i(i + 1)/2 + 1$. By the choice of the points x_i, we have that for $1 \le q \le s$ and $0 \le j \le i - 1$ the function $f^{(j)}(x)$ is monotonic and does not change sign on the interval $[x_{q-1}, x_q]$. Therefore, in view of the definition of D^1 we must have that $\Delta_q = D^1 \cap [x_{q-1}, x_q]$ is an interval. Hence, $D^1 = \bigcup_{q=1}^{s} \Delta_q$, a union of at most $(i + 1)i/2 + 1 \le (n + 1)n/2 + 1$ intervals.

It remains to estimate the length of each Δ_q. To this end, take any $x_1, x_2 \in \Delta_q$. By the construction of Δ_q, the numbers $f(x_1)$ and $f(x_2)$ have the same sign and satisfy the inequality $|f(x_i)| < \kappa b^{-t}$. Hence, $|f(x_1) - f(x_2)| < \kappa b^{-t}$. By the Mean Value Theorem, $|f(x_1) - f(x_2)| = |f'(\theta)(x_1 - x_2)|$. Hence $|x_1 - x_2| \le \kappa b^{-t}/|f'(\theta)|$. Since $\Delta_q \subset D^1$ is an interval, $\theta \in D^1$. Hence, $|f'(\theta)| \ge b^{\gamma t - (1+\gamma)\ell}$ and we obtain that $|x_1 - x_2| \le \kappa b^{-t} b^{-\gamma t + (1+\gamma)\ell} = \kappa b^{-(1+\gamma)(t-\ell)}$. This estimate together with the obvious equality $|\Delta_q| = \sup_{x_1, x_2 \in \Delta_q} |x_1 - x_2|$ implies that $|\Delta_q| \le \kappa b^{-(1+\gamma)(t-\ell)}$.

Now consider the case $i=1$. Recall that $f(x)=a_0+\mathbf{a}.\mathbf{f}(x)$. Then, by the definition of D^1 and (25), for $x\in D^1$ we get

$$b^{\gamma t-(1+\gamma)\ell}\le |f'(x)|=|\mathbf{a}.\mathbf{f}'(x)|\le c_1 n\max_{1\le i\le n}|a_i|. \tag{29}$$

Further, $(26)_{i=1}$ implies that $\inf_{x\in I_0}|f'(x)|\ge c_2\max_{1\le j\le n}|a_j|$. Therefore, f is monotonic on I_0 and D^1 is covered by a single interval Δ defined by the inequality $|f(x)|<\kappa b^{-t}$. Arguing as above and using (29) we get

$$\begin{aligned}|\Delta| &\le \frac{2\kappa b^{-t}}{\inf_{x\in I_0}|f'(x)|}\le\frac{2\kappa b^{-t}}{c_2\max_{1\le j\le n}|a_j|}\\ &\le \frac{2c_1 n\kappa b^{-t}}{c_2 b^{\gamma t-(1+\gamma)\ell}}=\frac{2c_1 n}{c_2}\times\kappa b^{-(1+\gamma)(t-\ell)}.\end{aligned}$$

Thus, by splitting Δ into smaller intervals if necessary, D^1 can be covered by at most $[\frac{2c_1n}{c_2}]+1$ intervals of length $\kappa b^{-(1+\gamma)(t-\ell)}$. ⊠

Proposition 2 *Let $I_0\subset I$ be a compact interval satisfying Property F and $\gamma=\gamma(\mathbf{r})$ be given by (28). Then there are constants $K_0>0$ and $0<\kappa_0<1$ such that for any $\mathbf{f}\in\mathcal{F}_n(I)$, any $\mathbf{r}\in\mathcal{R}_n$, $t\in\mathbb{N}$, $0\le\varepsilon<\gamma$, $b>1$ and $0<\kappa<\kappa_0$ the set*

$$D^2_{t,\varepsilon,\mathbf{r},b,\kappa,\mathbf{f}}=\left\{x\in I_0\ :\ \exists\ \mathbf{a}\in\mathbb{Z}^n\setminus\{0\}\ \text{and}\ a_0\in\mathbb{Z}\ \ \begin{array}{l}|a_0+\mathbf{a}.\mathbf{f}(x)|<\kappa\, b^{-t}\\ |\mathbf{a}.\mathbf{f}'(x)|<nc_1b^{(\gamma-\varepsilon)t}\\ |a_i|<b^{r_it}\end{array}\right\}.$$

can be covered by a collection $\mathcal{D}^2_{t,\varepsilon,\mathbf{r},b,\kappa,\mathbf{f}}$ of intervals such that

$$|\Delta|\le\delta_t\qquad\text{for all}\ \ \Delta\in\mathcal{D}^2_{t,\varepsilon,\mathbf{r},b,\kappa,\mathbf{f}} \tag{30}$$

and

$$\#\mathcal{D}^2_{t,\varepsilon,\mathbf{r},b,\kappa,\mathbf{f}}\le\frac{K_0\,(\kappa\, b^{-\varepsilon t})^\alpha}{\delta_t}\,, \tag{31}$$

where $\delta_t=\kappa\, b^{-t(1+\gamma-\varepsilon)}$ and $\alpha=\frac{1}{(n+1)(2n-1)}$.

Proposition 2 will be derived from a theorem due to Bernik, Kleinbock and Margulis using the ideas of [BBD02]. In what follows $|X|$ denotes the Lebesgue measure of a set $X\subset\mathbb{R}$. The following is a simplified version of Theorem 1.4 from [BKM01] that refines the results of [KM98].

Theorem 3 (Theorem 1.4 in [BKM01]) *Let $I\subset\mathbb{R}$ be an open interval, $x_0\in I$ and $\mathbf{f}:I\to\mathbb{R}^n$ be nondegenerate at x_0. Then there is an open interval $J\subset I$ centred at x_0 and $E_J>0$ such that for any real $\omega,K,T_1,\dots,T_n$ satisfying*

$$0<\omega\le 1,\quad T_1,\dots,T_n\ge 1,\quad K>0\quad\text{and}\quad \omega KT_1\cdots T_n\le\max_i T_i$$

the set

$$S(\omega,K,T_1,\dots,T_n):=\left\{x\in I\ :\ \exists\ \mathbf{a}\in\mathbb{Z}^n\setminus\{0\}\quad \begin{array}{l}\|\mathbf{a}.\mathbf{f}(\mathbf{x})\|<\omega\\ |\mathbf{a}.\mathbf{f}'(\mathbf{x})|<K\\ |a_i|<T_i\quad(1\le i\le n)\end{array}\right\}$$

satisfies

$$|S(\omega,K,T_1,\dots,T_n)\cap J| \ \le \ E_J\cdot\max\left(\omega,\left(\frac{\omega K T_1\cdots T_n}{\max_i T_i}\right)^{\frac{1}{n+1}}\right)^{\frac{1}{2n-1}}. \qquad (32)$$

We will also use the following elementary consequence of Taylor's formula.

Lemma 8 *Let $f:J\to\mathbb{R}$ be a C^2 function on an interval J. Let $\omega,K>0$ and $y\in J$ be such that $|f''(x)|<K^2/\omega$ for all $x\in J$ and*

$$|f(y)|<\omega/2 \qquad \text{and} \qquad |f'(y)|<K/2\,. \qquad (33)$$

Then $|f(x)|<\omega$ and $|f'(x)|<K$ for all $x\in J$ with $|x-y|<\omega/2K$.

Proof of Proposition 2. Fix any $\mathbf{f}\in\mathcal{F}_n(I)$. We will abbreviate $D^2_{t,\varepsilon,\mathbf{r},b,\kappa,\mathbf{f}}$ as D^2 and naturally assume that it is non-empty as otherwise there is nothing to prove. By (25), $\mathbf{f}$ is nondegenerate at any $x\in I_0$ and therefore Theorem 3 is applicable. Let $J=J(x)$ be the interval centred at x that arises from Theorem 3. Since I_0 is compact there is a finite cover of I_0 by intervals $J(x_1),\dots,J(x_s)$, where $s=s_{\mathbf{f}}$ depends on $\mathbf{f}$. Let $0<\kappa_0<1$ and $\kappa_0\le\min_{1\le i\le s}|I_0\cap J(x_i)|$. The existence of κ_0 is obvious because, $|I_0\cap J(x)|>0$ for each $x\in I_0$.

Let $0<\kappa<\kappa_0$, $\mathbf{r}\in\mathcal{R}_n$, $t\in\mathbb{N}$, $0\le\varepsilon<\gamma$, $b>1$ and let

$$\omega=2\kappa\, b^{-t}, \qquad K=2nc_1b^{(\gamma-\varepsilon)t} \qquad \text{and} \qquad T_i=b^{r_it}\quad(1\le i\le n). \qquad (34)$$

Note that since $\varepsilon<\gamma$ and $c_1>1$ we have that $K>2$. Also note that $\omega<2\kappa$. For each $i\in\{1,\dots,s\}$ define the interval $J_i=(a_i+\omega/2K,b_i-\omega/2K)$, where $[a_i,b_i]$ is the intersection of I_0 and the closure of $J(x_i)$. Since $\kappa<\kappa_0\le|I_0\cap J(x_i)|$, $\omega<2\kappa$ and $K>2$, we have that $J_i\neq\emptyset$ for each i. Let

$$\tilde D^2=\bigcup_{1\le i\le s}\ \bigcup_{y\in D^2\cap J_i}\big(y-\omega/2K,y+\omega/2K\big). \qquad (35)$$

Our goal now is to use Lemma 8 with $f(x)=a_0+\mathbf{a}.\mathbf{f}(x)$ in order to show that

$$\tilde D^2\subset\bigcup_{1\le i\le s}S(\omega,K,T_1,\dots,T_n)\cap J(x_i). \qquad (36)$$

In view of the definitions of D^2 and $S(\omega,K,T_1,\dots,T_n)$ and the choice of parameters (34), inequalities (33) hold for every $y\in D^2$. Further, by (25), the inequalities $|a_i|<b^{r_it}$ and the fact that $r_i\le\gamma$ for all i implied by (28), we get that

$$|f''(x)|\le nc_1\max_{1\le j\le n}|a_j|\le nc_1\max_{1\le j\le n}b^{r_it}\le nc_1b^{\gamma t}. \qquad (37)$$

Next, $K^2/\omega=\frac{1}{2}n^2c_1^2\kappa^{-1}b^{2(\gamma-\varepsilon)t}b^t>nc_1b^{\gamma t}$ because $\varepsilon\le\gamma\le1$, $c_1>1$ and $\kappa<1$. Therefore, by (37), we have that $|f''(x)|\le K^2/\omega$ for all $x\in I_0$. Thus, Lemma 8 is applicable and for $1\le i\le s$ we have that $\{x:|x-y|<\omega/2K\}\subset S(\omega,K,T_1,\dots,T_n)\cap J(x_i)$ each $y\in D^2\cap J_i$. This proves (36).

Next, by Theorem 3, condition $r_1+\dots+r_n=1$ and (36) we conclude that

$$|\tilde D^2|\le E_{\mathbf{f}}\cdot\big(2nc_1\kappa b^{-\varepsilon t}\big)^{\alpha} \qquad (38)$$

where $E_{\mathbf{f}} = s\max_{1\le i\le s} E_{J(x_i)}$. By (35), $\tilde{D}^2$ can be written as a union of disjoint intervals of length $\ge \omega/K = 2(nc_1)^{-1}\kappa\, b^{-t(1+\gamma-\varepsilon)} > (nc_1)^{-1}\delta_t$. By splitting some of these intervals if necessary, we get a collection $\tilde{\mathcal{D}}^2$ of disjoint intervals Δ such that $\frac{1}{c_1 n}\delta_t \le |\Delta| \le \delta_t$. Let

$$K_0 = \max_{\mathbf{f}\in\mathcal{F}_n(I)} \max\{4s_{\mathbf{f}}, (4nc_1)^{1+\alpha}E_{\mathbf{f}}\}.$$

Then, by (38) and the above inequality, we get

$$\#\tilde{\mathcal{D}}^2 \le \frac{E_{\mathbf{f}}\cdot\left(4nc_1\kappa b^{-\varepsilon t}\right)^{\alpha}}{\frac{1}{c_1 n}\delta_t} \le \frac{K_0\,(\kappa\, b^{-\varepsilon t})^{\alpha}}{2\delta_t}. \tag{39}$$

Let $\mathcal{D}^2$ be the collection of all the intervals in $\tilde{\mathcal{D}}^2$ together with the $2s$ intervals $[a_i, a_i+\omega/2K]$ and $[b_i-\omega/2K, b_i]$ $(1\le i\le s)$. It is easily seen that $2s$ less than or equal to the right hand side of (39). Then, by (39) and the definition of $\mathcal{D}^2$, we get (30) and (31). Also, by construction, we see that $\mathcal{D}^2$ is a cover of D^2. The proof is thus complete.

$\boxtimes$

5 A Cantor sets framework

Let $R\ge 2$ be an integer. Given a collection $\mathcal{I}$ of compact intervals in $\mathbb{R}$, let $\frac{1}{R}\mathcal{I}$ denote the collection of intervals obtained by dividing each interval in $\mathcal{I}$ into R equal closed subintervals. For example, for $R=3$ and $\mathcal{I}=\{[0,1]\}$ we have that $\frac{1}{R}\mathcal{I} = \big\{[0,\frac{1}{3}],[\frac{1}{3},\frac{2}{3}],[\frac{2}{3},1]\big\}$. Let $I_0\subset\mathbb{R}$ be a compact interval. The sequence $(\mathcal{I}_q)_{q\ge 0}$ will be called *an R-sequence in I_0* if

$$\mathcal{I}_0 = \{I_0\} \qquad \text{and} \qquad \mathcal{I}_q \subset \tfrac{1}{R}\mathcal{I}_{q-1} \quad \text{for } q\ge 1\,. \tag{40}$$

Naturally, I_q will denote any interval from the collection $\mathcal{I}_q$. Observe that

$$|I_q| = R^{-q}|I_0| \qquad \text{for } q\ge 0. \tag{41}$$

By definition, given $I_q\in\mathcal{I}_q$ with $q\ge 1$, there is a unique interval $I_{q-1}\in\mathcal{I}_{q-1}$ such that $I_q\subset I_{q-1}$; this interval I_{q-1} will be called the *precursor* of I_q. Obviously it is independent of the choice of R-sequence $(\mathcal{I}_q)_{q\ge 0}$ with $I_q\in\mathcal{I}_q$.

The limit set of $(\mathcal{I}_q)_{q\ge 0}$ is defined to be

$$\mathcal{K}(\mathcal{I}_q) \stackrel{\text{def}}{=} \bigcap_{q\ge 0}\ \bigcup_{I_q\in\mathcal{I}_q} I_q. \tag{42}$$

This is *a Cantor type set*. The classical middle third Cantor set can be constructed this way in an obvious manner with $R=3$ and $I_0=[0,1]$. Theorem 2 will be proved by finding suitable Cantor type sets $\mathcal{K}(\mathcal{I}_q)$. Note that if $\mathcal{I}_q\ne\emptyset$ for all q so that $\mathcal{I}_q$ is genuinely an infinite sequence, then $\mathcal{K}(\mathcal{I}_q)\ne\emptyset$. However, ensuring that $\mathcal{K}(\mathcal{I}_q)$ is large requires better understanding of the sets $\mathcal{I}_q$. There are various techniques in fractal geometry that are geared towards this task – see [Fal03]. We shall use a recent powerful result of Badziahin and Velani [BV11] restated below using our notation. Given $q\in\mathbb{N}$ and an interval J, let

$$\widehat{\mathcal{I}}_q \stackrel{\text{def}}{=} \left(\tfrac{1}{R}\mathcal{I}_{q-1}\right)\setminus\mathcal{I}_q\,, \qquad \widehat{\mathcal{I}}_q\sqcap J \stackrel{\text{def}}{=} \{I_q\in\widehat{\mathcal{I}}_q : I_q\subset J\}$$

and

$$d_q(\mathcal{I}_q) \;=\; \min_{\{\widehat{\mathcal{I}}_{q,p}\}} \sum_{p=0}^{q-1} \left(\frac{4}{R}\right)^{q-p} \max_{I_p \in \mathcal{I}_p} \#\big(\widehat{\mathcal{I}}_{q,p} \sqcap I_p\big)\ , \tag{43}$$

where the minimum is taken over all partitions $\{\widehat{\mathcal{I}}_{q,p}\}_{p=0}^{q-1}$ of $\widehat{\mathcal{I}}_q$, that is $\widehat{\mathcal{I}}_q = \bigcup_{p=0}^{q-1} \widehat{\mathcal{I}}_{q,p}$. Also define

$$d(\mathcal{I}_q) = \sup_{q>0} d_q(\mathcal{I}_q).$$

Theorem 4 (Theorem 4 in [BV11]) *Let $R \geq 4$ be an integer, I_0 be a compact interval in $\mathbb{R}$ and $(\mathcal{I}_q)_{q\geq 0}$ be an R-sequence in I_0. If $d(\mathcal{I}_q) \leq 1$ then*

$$\dim \mathcal{K}(\mathcal{I}_q) \;\geq\; \left(1 - \frac{\log 2}{\log R}\right). \tag{44}$$

Let $M > 1$, $X \subset \mathbb{R}$ and I_0 be a compact interval. We will say that X is *M-Cantor rich in I_0* if for any $\varepsilon > 0$ and any integer $R \geq M$ there exists an R-sequence $(\mathcal{I}_q)_{q\geq 0}$ in I_0 such that $\mathcal{K}(\mathcal{I}_q) \subset X$ and $d(\mathcal{I}_q) \leq \varepsilon$. We will say that X is *Cantor rich in I_0* if it is *M-Cantor rich in I_0* for some M. We will say that X is *Cantor rich* if it is *Cantor rich in I_0* for some compact interval I_0. The following statement readily follows from Theorem 4 and our definitions.

Theorem 5 *Any Cantor rich set X satisfies* $\dim X = 1$.

We now proceed with a discussion of the intersections of Cantor rich sets. To some extent this already appears in [BV11, Theorem 5] and in [BPV11]. First we prove the following auxiliary statement.

Lemma 9 *Let $(\mathcal{I}_q^j)_{q\geq 0}$ be a family of R-sequences in I_0 indexed by j. Given $q \in \mathbb{Z}_{\geq 0}$, let $\mathcal{J}_q = \bigcap_j \mathcal{I}_q^j$. Then $(\mathcal{J}_q)_{q\geq 0}$ is an R-sequence in I_0 such that*

$$\widehat{\mathcal{J}}_q \subset \textstyle\bigcup_j \widehat{\mathcal{I}}_q^j \qquad \text{for all } q \geq 0 \tag{45}$$

and

$$\mathcal{K}(\mathcal{J}_q) \subset \textstyle\bigcap_j \mathcal{K}(\mathcal{I}_q^j). \tag{46}$$

Proof. The validity of (40) for $(\mathcal{J}_q)_{q\geq 0}$ follows from the uniqueness of the precursor of an interval in any R-sequence from that sequence and the fact that $\mathcal{I}_0^j = \{I_0\}$ for all j, which means that $\mathcal{J}_0 = \bigcap_j \mathcal{I}_0^j = \{I_0\}$. Thus, $(\mathcal{J}_q)_{q\geq 0}$ is truly an R-sequence. The inclusion (45) is obvious for $q = 0$ for both sides of the inclusion are empty sets in this case. To see (45) for $q > 0$, observe that $\mathcal{J}_{q-1} \subset \mathcal{I}_{q-1}^j$ and this implies that $\frac{1}{R}\mathcal{J}_{q-1} \subset \frac{1}{R}\mathcal{I}_{q-1}^j$ for each j. Then we have

$$\begin{aligned} \widehat{\mathcal{J}}_q \;=\; \tfrac{1}{R}\mathcal{J}_{q-1} \setminus \mathcal{J}_q \;&=\; \tfrac{1}{R}\mathcal{J}_{q-1} \setminus \textstyle\bigcap_j \mathcal{I}_q^j \;=\; \textstyle\bigcup_j \big(\tfrac{1}{R}\mathcal{J}_{q-1} \setminus \mathcal{I}_q^j\big) \\ &\subset\; \textstyle\bigcup_j \big(\tfrac{1}{R}\mathcal{I}_{q-1}^j \setminus \mathcal{I}_q^j\big) \;=\; \textstyle\bigcup_j \widehat{\mathcal{I}}_q^j. \end{aligned}$$

Finally, by the inclusion $\mathcal{J}_q \subset \mathcal{I}_q^j$, we have that $\bigcup J_q \subset \bigcup I_q^j$ for each pair of j and q, where the union is taken over $J_q \in \mathcal{J}_q$ and $I_q^j \in \mathcal{I}_q^j$ respectively. Hence, by (42), we have that $\mathcal{K}(\mathcal{J}_q) \subset \mathcal{K}(\mathcal{I}_q^j)$ for all j, whence (46) now follows. $\boxtimes$

Theorem 6 *Let I_0 be a compact interval. Then any countable intersection of M-Cantor rich sets in I_0 is M-Cantor rich in I_0. In particular, any finite intersection of Cantor rich sets in I_0 is Cantor rich in I_0.*

Proof. Let $\{X_j\}_{j\in\mathbb{N}}$ be a collection of M-Cantor rich sets in I_0. Let $\varepsilon > 0$. Then, by definition, for each $j \in \mathbb{N}$ and $R \geq M$ there is an R-sequence $(\mathcal{I}_q^j)_{q\geq 0}$ in I_0 such that $\mathcal{K}(\mathcal{I}_q^j) \subset X_j$ and $d_q(\mathcal{I}_q^j) \leq \varepsilon 2^{-j}$ for all $q > 0$. By (43), for each j and $q > 0$ there exists a partition $\{\widehat{\mathcal{I}}_{q,p}^j\}_{p=0}^{q-1}$ of $\widehat{\mathcal{I}}_q^j$ such that

$$\sum_{p=0}^{q-1} \left(\frac{4}{R}\right)^{q-p} \max_{I_p\in\mathcal{I}_p^j} \#\big(\widehat{\mathcal{I}}_{q,p}^j \sqcap I_p\big) \ \leq \varepsilon 2^{-j}. \tag{47}$$

For $q \in \mathbb{Z}_{\geq 0}$ define $\mathcal{J}_q = \bigcap_{j\in\mathbb{N}} \mathcal{I}_q^j$ and $\widehat{\mathcal{J}}_{q,p} \stackrel{\text{def}}{=} \widehat{\mathcal{J}}_q \cap \bigcup_{j\in\mathbb{N}} \widehat{\mathcal{I}}_{q,p}^j$. Since $\widehat{\mathcal{I}}_q^j = \bigcup_{p=0}^{q-1} \widehat{\mathcal{I}}_{q,p}^j$ for each j, by (45), we have that $\widehat{\mathcal{J}}_q = \bigcup_{p=0}^{q-1} \widehat{\mathcal{J}}_{q,p}$, where $q > 0$. Then, for each $q > 0$ we get that

$$\sum_{p=0}^{q-1} \left(\frac{4}{R}\right)^{q-p} \max_{J_p\in\mathcal{J}_p} \#\big(\widehat{\mathcal{J}}_{q,p} \sqcap J_p\big) \leq \sum_{j=1}^{\infty}\sum_{p=0}^{q-1} \left(\frac{4}{R}\right)^{q-p} \max_{I_p\in\mathcal{I}_p^j} \#\big(\widehat{\mathcal{I}}_{q,p}^j \sqcap I_p\big).$$

This inequality together with (47) and the definition of $d(\mathcal{J}_q)$ implies that $d(\mathcal{J}_q) \leq \varepsilon$. By (46) and the fact that $\mathcal{K}(\mathcal{I}_q^j) \subset X_j$ for each j, we have that $\mathcal{K}(\mathcal{J}_q) \subset \bigcap_j X_j$. Thus the intersection $\bigcap_j X_j$ meets the definition of M-Cantor rich sets and the proof is complete. ⊠

The winning sets in the sense of Schmidt have been used a lot to investigate various sets of badly approximable points. Hence we suggest the following

Problem: Verify that any α-winning set in $\mathbb{R}$ as defined by Schmidt [Sch80] is M-Cantor rich and find a bound on M in terms of α.

6 Proof of Theorem 2

The following proposition is a key step to establishing Theorem 2. We will use the Vinogradov symbol $\ll$ to simplify the calculations. The expression $X \ll Y$ will mean that $X \leq CY$ for some $C > 0$, which only depends on n, the family of maps $\mathcal{F}_n(I)$ from Theorem 2 and the interval I_0 occurring in Property F.

Proposition 3 *Let $\mathcal{F}_n(I)$ be as in Theorem 2, $I_0 \subset I$ be a compact interval satisfying Property F and $\sigma = 1 - (2n)^{-4}$. Then there are constants $R_0 \geq 4$ and $m_0 \geq 4$ such that for any $\mathbf{f} \in \mathcal{F}_n(I)$, $\mathbf{r} \in \mathcal{R}_n$ and any integers $m \geq m_0$ and $R \geq R_0$, there exists an R-sequence $(\mathcal{I}_q)_{q\geq 0}$ in I_0 such that*

(i) *for any $t \in \mathbb{N}$ and any $I_{t+m} \in \mathcal{I}_{t+m}$ we have that*

$$\delta\big(g^t G_x \mathbb{Z}^{n+1}\big) \geq 1 \qquad \text{for all } x \in I_{t+m}; \tag{48}$$

where $g^t = g^t_{\mathbf{r},b}$ is given by (13) with $b^{1+\gamma} = R$, $\gamma = \gamma(\mathbf{r})$ and $G_x = G(\kappa; \mathbf{f}(x))$ is given by (12) with $\kappa = R^{-m}$;

(ii) *if* $q \le m$ *then* $\#\widehat{\mathcal{I}}_q = 0$;

(iii) *if* $q = t + m$ *for some* $t \in \mathbb{N}$ *then* $\widehat{\mathcal{I}}_q$ *can be written as the union* $\widehat{\mathcal{I}}_q = \bigcup_{p=0}^{q-1} \widehat{\mathcal{I}}_{q,p}$ *such that for integers* $p = t + 3 - 2\ell$ *with* $0 \le \ell \le \ell_t = [t/2n] + 1$ *and* $I_p \in \mathcal{I}_p$ *we have that*

$$\#(\widehat{\mathcal{I}}_{q,p} \sqcap I_p) \ \ll\ R^{\frac{1+\lambda}{2}(q-p) - \frac{1-\lambda}{2}m + 3}, \tag{49}$$

$$\#\widehat{\mathcal{I}}_{q,0} \ll R^{\sigma q} \tag{50}$$

and $\widehat{\mathcal{I}}_{q,p} = \emptyset$ *for all other* $p < q$*, where* $\lambda = \lambda(\mathbf{r})$ *is given by* (19)*.*

Proof. Define $\mathcal{I}_0 = \{I_0\}$ and $\mathcal{I}_q = \frac{1}{R}\mathcal{I}_{q-1}$ for $1 \le q \le m$. Then conditions (i) and (iii) are irrelevant, while (ii) is obvious. Continuing by induction, let $q = t + m$ with $t \ge 1$ and let us assume that $\mathcal{I}_{q'}$ with $q' < q$ are given and satisfy conditions (i)–(iii). Define $\mathcal{I}_q$ to be the collection of intervals from $\frac{1}{R}\mathcal{I}_{q-1}$ that satisfy (48). By construction, (i) holds, (ii) is irrelevant and we only need to verify condition (iii). We shall assume that $\widehat{\mathcal{I}}_q \ne \emptyset$ as otherwise (iii) is obvious. By construction, $\widehat{\mathcal{I}}_q$ consists of intervals I_q such that $\delta(g^t G_x \mathbb{Z}^{n+1}) < 1$ for some $x \in I_q$. Recall that this is equivalent to the existence of $(a_0, \mathbf{a}) \in \mathbb{Z}^{n+1}$ with $\mathbf{a} \ne \mathbf{0}$ satisfying the system (24). We shall use Propositions 1 and 2 and Lemma 5 to estimate the number of these intervals I_q. Before we proceed with the estimates note that, by (25) and (28), the validity of (24) implies that $|\mathbf{a}.\mathbf{f}'(x)| \le nc_1 \max_{1\le j\le n} |a_j| \le nc_1 \max_{1\le j\le n} b^{r_j t} = nc_1 b^{\gamma t}$. Thus,

$$\forall\ x \in I_0 \qquad \delta(g^t G_x \mathbb{Z}^{n+1}) < 1 \quad \Rightarrow \quad |\mathbf{a}.\mathbf{f}'(x)| \le nc_1 b^{\gamma t}. \tag{51}$$

The arguments split into 2 cases depending on the size of t as follows. Throughout we assume that $R_0^{-m_0} < \kappa_0$, where κ_0 is as in Proposition 2

Case 1: $t \le 2nm$. In this case let $\widehat{\mathcal{I}}_{q,0} = \widehat{\mathcal{I}}_q$ and $\widehat{\mathcal{I}}_{q,p} = \emptyset$ for $0 < p < q$. Then, the only thing we need to verify is (50). Let $\varepsilon = 0$. Then, by (51), we have that

$$\big\{x \in I_0 : \delta(g^t G_x \mathbb{Z}^{n+1}) < 1\big\} = D_0^2, \tag{52}$$

where $D_0^2 = D^2_{t,\varepsilon,\mathbf{r},b,\kappa,\mathbf{f}}$ (with $\varepsilon = 0$) as defined in Proposition 2. Hence, $\#\widehat{\mathcal{I}}_{q,0}$ is bounded by the number intervals in $\frac{1}{R}\mathcal{I}_{q-1}$ that intersect an interval from the corresponding collection $\mathcal{D}_0^2$ of intervals arising from Proposition 2. By (41), the intervals in $\frac{1}{R}\mathcal{I}_{q-1}$ are of length $R^{-q}|I_0|$. By (30), the intervals from $\mathcal{D}_0^2$ have length $\le \delta_t = \kappa\, b^{-t(1+\gamma)}$. Hence, each interval from $\mathcal{D}_0^2$ can intersect at most $\delta_t/(R^{-q}|I_0|) + 2 = \kappa\, b^{-t(1+\gamma)} R^q |I_0|^{-1} + 2$ intervals from $\frac{1}{R}\mathcal{I}_{q-1}$. Since $b^{1+\gamma} = R$, $\kappa = R^{-m}$ and $q = t + m$, we have that $\delta_t/(R^{-q}|I_0|) = |I_0|^{-1}$. Hence each interval from $\mathcal{D}_0^2$ can intersect $\ll \delta_t R^q$ intervals from $\frac{1}{R}\mathcal{I}_{q-1}$. Then, by (31), we get

$$\#\widehat{\mathcal{I}}_{q,0} \ \ll \delta_t R^q \times \frac{K_0\, \kappa^\alpha}{\delta_t} \ll R^q \times \kappa^\alpha. \tag{53}$$

Using $q = t + m$, $\kappa = R^{-m}$ and $t \le 2nm$ we obtain from (53) that

$$\#\widehat{\mathcal{I}}_{q,0} \ \ll R^{t+m} \times (R^{-m})^\alpha \le R^{\left(1-\frac{\alpha}{2n+1}\right)(t+m)} = R^{\left(1-\frac{\alpha}{2n+1}\right)q}. \tag{54}$$

Recall from Proposition 2 that $\alpha = \frac{1}{(n+1)(2n-1)}$. Consequentially $\sigma \ge 1 - \frac{\alpha}{2n+1}$ and (54) implies (50).

Case 2: $t > 2nm$. Let $\varepsilon = (2n)^{-1}$. Since $\sum_i r_i = 1$ and $\gamma = \max\{r_1, \dots, r_n\}$, we have that $\gamma \geq 1/n$. Hence $\varepsilon < \gamma$. Assume that $R > nc_1$. Then, by (51) and the choice of ε, for any $x \in I_0$ such that $\delta(g^t G_x \mathbb{Z}^{n+1}) < 1$ we have that either

$$|\mathbf{a}.\mathbf{f}'(x)| < nc_1 b^{(\gamma-\varepsilon)t}$$

or for some $\ell \in \mathbb{Z}$ with $0 \leq \ell \leq \ell_t = [t/2n] + 1$

$$b^{\gamma t-(1+\gamma)\ell} \leq \ |\mathbf{a}.\mathbf{f}'(x)| < b^{\gamma t-(1+\gamma)(\ell-1)}.$$

Then, once again using the equivalence of $\delta(g^t G_x \mathbb{Z}^{n+1}) < 1$ to the existence of $(a_0, \mathbf{a}) \in \mathbb{Z}^{n+1}$ with $\mathbf{a} \neq \mathbf{0}$ satisfying (24), we write that

$$\left\{x \in I_0 : \delta(g^t G_x \mathbb{Z}^{n+1}) < 1\right\} = \bigcup_{\ell=0}^{\ell_t} \ \bigcup_{\mathbf{a}\in\mathbb{Z}^n\setminus\{\mathbf{0}\}} \ \bigcup_{a_0\in\mathbb{Z}} D_\ell^1(a_0, \mathbf{a}) \cup D^2, \tag{55}$$

where

$$D_\ell^1(a_0, \mathbf{a}) = D_{t,\ell,\mathbf{r},b,\kappa,\mathbf{f}}^1(a_0, \mathbf{a}) \qquad \text{and} \qquad D^2 = D_{t,\varepsilon,\mathbf{r},b,\kappa,\mathbf{f}}^2$$

as defined in Propositions 1 and 2 respectively.

By definition, intervals in $\widehat{\mathcal{I}}_q$ are characterised by having a non-empty intersection with the left hand side of (55). We now use the right hand side of (55) to define the subcollections $\widehat{\mathcal{I}}_{q,p}$ of $\widehat{\mathcal{I}}_q$. More precisely, for $p = t + 3 - 2\ell$ with $0 \leq \ell \leq \ell_t$ let $\widehat{\mathcal{I}}_{q,p}$ consist of the intervals $I_q \in \widehat{\mathcal{I}}_q$ that intersect $D_\ell^1(a_0, \mathbf{a})$ for some $\mathbf{a} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$ and $a_0 \in \mathbb{Z}$. Next, let $\widehat{\mathcal{I}}_{q,0}$ consist of the intervals $I_q \in \widehat{\mathcal{I}}_q$ that intersect D^2. Finally, define $\widehat{\mathcal{I}}_{q,p} = \emptyset$ for all other $p < q$. By (52), it is easily seen that $\widehat{\mathcal{I}}_q = \bigcup_{p=0}^{q-1} \widehat{\mathcal{I}}_{q,p}$. It remains to verify (49) and (50).

• *Verifying (50).* This is very much in line with Case 1. The goal is to count the number intervals in $\frac{1}{R}\mathcal{I}_{q-1}$ that intersect some interval from the collection $\mathcal{D}^2$ arising from Proposition 2. By (41), the intervals in $\frac{1}{R}\mathcal{I}_{q-1}$ are of length $R^{-q}|I_0|$. By (30), the intervals from $\mathcal{D}^2$ have length $\leq \delta_t = \kappa\, b^{-t(1+\gamma-\varepsilon)}$. Hence, each interval from $\mathcal{D}^2$ can intersect at most $\delta_t/(R^{-q}|I_0|) + 2 \ll \delta_t R^q$ intervals from $\frac{1}{R}\mathcal{I}_{q-1}$. Then, by (31), we get

$$\#\widehat{\mathcal{I}}_{q,0} \ \ll \delta_t R^q \times \frac{K_0\,(\kappa\, b^{-\varepsilon t})^\alpha}{\delta_t} \ll R^q \times (\kappa\, b^{-\varepsilon t})^\alpha.$$

Using $\kappa = R^{-m}$, $b^{1+\gamma} = R$, $q = t + m$ and $0 < \gamma \leq 1$, we obtain that

$$\#\widehat{\mathcal{I}}_{q,0} \ \ll R^q \times (R^{-m}\, R^{-\varepsilon/(1+\gamma)t})^\alpha \leq R^q R^{-\frac{\varepsilon\alpha}{2}(t+m)} = R^{(1-\varepsilon\alpha/2)q}. \tag{56}$$

Once again using the value of α from Proposition 2 we verify that $\sigma \geq 1 - \frac{1}{2}\varepsilon\alpha$ and so (56) implies (50) as required.

• *Verifying (49).* Let $p = t + 3 - 2\ell$ with $0 \leq \ell \leq \ell_t$ and $I_p \in \mathcal{I}_p$. Let $S(I_p)$ be the set of points $(a_0, \mathbf{a}) \in \mathbb{Z}^{n+1}$ with $\mathbf{a} \neq \mathbf{0}$ such that $D_\ell^1(a_0, \mathbf{a}) \cap I_p \neq \emptyset$. By Proposition 1, for every $(a_0, \mathbf{a}) \in S(I_p)$ any interval in $\mathcal{D}_\ell^1(a_0, \mathbf{a})$ is of length

$$\leq \kappa b^{-(1+\gamma)(t-\ell)} = R^{-m} R^{-(t-\ell)} = R^{-(t+m-\ell)} = R^{q-\ell}.$$

as $\kappa = R^{-m}$, $b^{1+\gamma} = R$ and $q = t+m$. Then, by (41), any interval from $\mathcal{D}^1_\ell(a_0, \mathbf{a})$ intersects $\ll R^\ell$ intervals from $\frac{1}{R}\mathcal{I}_{q-1}$. By Proposition 1, $\#\mathcal{D}^1_\ell(a_0, \mathbf{a}) \ll 1$. Hence,

$$\#(\widehat{\mathcal{I}}_{q,p} \sqcap I_p) \ll \#S(I_p) \times R^\ell \tag{57}$$

and our main concern becomes to obtain a bound for $\#S(I_p)$. We shall prove that

$$\#S(I_p) \ll R^{\frac{\tau}{1+\gamma}+\lambda(m+\ell-1)}. \tag{58}$$

Armed with this estimate establishing (49) and thus completing our task becomes simple. Indeed, using (57) and (58) gives

$$\#(\widehat{\mathcal{I}}_{q,p} \sqcap I_p) \ll R^{\frac{1+\lambda}{2}(2\ell+m-3)-\frac{1-\lambda}{2}m+\frac{3+\lambda}{2}+\frac{\tau}{1+\gamma}},$$

which implies (49) upon observing that $2\ell + m - 3 = q - p$ and $\frac{3+\lambda}{2} + \frac{\tau}{1+\gamma} < 3$.

Proof of (58). We assume that $S(I_p) \neq \emptyset$ as otherwise (58) is trivial. The proof will be split into several relatively simple steps.

Step 1: We show that for any $(a_0, \mathbf{a}) \in S(I_p)$ and any $x \in I_p$ we have

$$|a_0 + \mathbf{a}.\mathbf{f}(x)| \ll b^{-t+(1+\gamma)(\ell-2)} \qquad \text{and} \qquad |\mathbf{a}.\mathbf{f}'(x)| \ll b^{\gamma t-(1+\gamma)(\ell-1)}. \tag{59}$$

Both inequalities are established the same way, so we give detailed arguments only for the right hand side estimate. To this end, fix any $(a_0, \mathbf{a}) \in S(I_p)$ and let $x_0 \in D^1_\ell(a_0, \mathbf{a}) \cap I_p$. To simplify notation define $f(x) = a_0 + \mathbf{a}.\mathbf{f}(x)$. By the Mean Value Theorem, for any $x \in I_p$ we have

$$|f'(x)| = |f'(x_0) + f''(\tilde{x}_0)(x - x_0)| \leq |f'(x_0)| + |f''(\tilde{x}_0)(x - x_0)|, \tag{60}$$

where $\tilde{x}_0$ is a point between x and x_0. By the definition of $D^1_\ell(a_0, \mathbf{a})$, we have that $|f'(x_0)| \leq b^{\gamma t-(1+\gamma)(\ell-1)}$. By (25), we get $|f''(\tilde{x}_0)| \leq nc_1 \max_{1\leq j\leq n} |a_j|$. Further, since $\gamma = \max\{r_1, \ldots, r_n\}$, we obtain that $\max_{1\leq j\leq n} |a_j| \leq \max_{1\leq j\leq n} b^{r_j t} \leq b^{\gamma t}$. Hence, $|f''(\tilde{x}_0)| \leq nc_1 b^{\gamma t}$. Substituting the estimates for $|f'(x_0)|$ and $|f''(\tilde{x}_0)|$ into (60) and using the inequity $|x - x_0| \leq |I_p| = R^{-p}|I_0| = R^{-(t+3-2\ell)}|I_0|$ implied by (41), we get

$$|f'(x)| \leq b^{\gamma t-(1+\gamma)(\ell-1)} + nc_1 b^{\gamma t} \times R^{-(t+3-2\ell)}|I_0|.$$

Since $b^{1+\gamma} = R$, we have that

$$|f'(x)| \ll b^{\gamma t-(1+\gamma)(\ell-1)} + b^{\gamma t-(1+\gamma)(t+3-2\ell)}. \tag{61}$$

Since $\ell \leq \ell_t < t/4 + 1$, one easily verifies that $(t + 3 - 2\ell) > (\ell - 1)$. Therefore (61) implies the right hand side of (59). As we mentioned, the proof of the left hand side is similar.

Step 2: Now we utilize (59) to show that $\operatorname{rank} S(I_p) \leq n - z$. First of all, observe that if $(a_0, \mathbf{a}) \in S(I_p)$, where $\mathbf{a} = (a_1, \ldots, a_n)$, then $|a_j| < b^{r_j t} = 1$ whenever $r_j = 0$. Since $a_j \in \mathbb{Z}$ in this case, we have that

$$\forall\ (a_0, \mathbf{a}) \in S(I_p) \qquad a_j = 0 \qquad \text{whenever} \quad r_j = 0. \tag{62}$$

Let $J = \{j : r_j \neq 0\}$ and $\overline{J} = \{1, \ldots, n\} \setminus J$. Note that J contains exactly $n - z > 0$ elements, where $z = z(\mathbf{r})$ is the number of zeros in $\mathbf{r}$. Let J_0 be the

subset of J obtained by removing the smallest index j_0 such that $r_{j_0} = \gamma(\mathbf{r})$. Note that if $\mathbf{r}$ has only one non-zero component then $J_0 = \emptyset$. Now, using (62) and (59) we obtain that every $(a_0, \mathbf{a}) \in S(I_p)$ satisfies the system

$$\left\{\begin{array}{rcll} |a_0 + \sum_{j\in J} a_j f_j(x)| & \ll & b^{-t+(1+\gamma)(\ell-2)}, & \\ |\sum_{j\in J} a_j f_j'(x)| & \ll & b^{\gamma t-(1+\gamma)(\ell-1)}, & \\ |a_j| & < & b^{r_j t} & (j \in J_0), \\ a_j & = & 0 & (j \in \overline{J}). \end{array}\right. \tag{63}$$

where $x \in I_p$. Let $\mathbf{B}_{p,x}$ denote the set of $(a_0, a_1, \ldots, a_n) \in \mathbb{R}^{n+1}$ satisfying (63). Then, $S(I_p) \subset \mathbf{B}_{p,x}$. Clearly, $\mathbf{B}_{p,x}$ is a convex body lying over the $n-z+1$ dimensional linear subspace of $\mathbb{R}^{n+1}$ given by the equations $a_j = 0$ for $j \in \overline{J}$. Using (25) one can easily see that the $n-z+1$-dimensional volume of $\mathbf{B}_{p,x}$ is

$$\ll \frac{1}{c_0} \times b^{-t+(1+\gamma)(\ell-2)} \times b^{\gamma t-(1+\gamma)(\ell-1)} \times \prod_{i\in J_0} b^{r_i t} = \frac{1}{c_0} b^{-(1+\gamma)} = \frac{1}{c_0 R}.$$

Thus if R is sufficiently large, $\mathrm{vol}_{n-z+1}(\mathbf{B}_{p,x}) < (n-z+1)!^{-1}$ and so, by Lemma 2, we have that $\operatorname{rank} S(I_p) \leq n-z$ as claimed at the start of Step 2.

Step 3: Finally, we obtain (58). To this end, let Γ denote the $\mathbb{Z}$-span of $S(I_p)$. Since $\operatorname{rank} S(I_p) \leq n-z$, we have that $\operatorname{rank} \Gamma \leq n-z$. Discarding the second inequality from (63) and estimating the constant arising from the Vinogradov symbol by R we obtain that the points $(a_0, \mathbf{a}) \in S(I_p)$ satisfy the system

$$\left\{\begin{array}{rcll} |a_0 + \sum_{j=1}^n a_j f_j(x)| & < & b^{-t+(1+\gamma)(\ell-1)}, & \\ |a_i| & < & b^{r_i t} & (1 \leq i \leq n) \end{array}\right. \tag{64}$$

provided that R is sufficiently large. This can be equivalently written as

$$S(I_p) \subset g^t G_x \Gamma \cap \Pi(b,u) \qquad \text{with} \qquad u = (1+\gamma)(m+\ell-1), \tag{65}$$

where $\Pi(b,u)$ is defined by (18). Note that $0 \leq \tau(\mathbf{r}) \leq 1/n \leq \gamma(\mathbf{r}) \leq 1$. Therefore $\lambda(1+\gamma) = (1+\gamma)/(1+\tau) \in [1,2]$. Hence $(m+\ell-1) \leq \lambda u \leq 2(m+\ell-1)$. Since $t > 2nm$, $m \geq 4$ and $\ell - 1 \leq t/2n$, one can easily see that $1 < \lambda u < t$. Hence $1 \leq t - [\lambda u] < t$. Take $x \in I_{t-[\lambda u]} \cap I_p$ for an appropriate interval $I_{t-[\lambda u]}$. By induction, (48) holds when t is replaced by $t - [\lambda u]$. This verifies (20) with $\Lambda = G_x\Gamma$. Clearly, $\operatorname{rank}\Lambda = \operatorname{rank}\Gamma \leq n-z$. Hence, by Lemma 5 and (65), we obtain that $\#S(I_p) \ll b^\tau b^{\lambda u}$. Now (58) readily follows upon substituting $b = R^{1/(1+\gamma)}$ and $u = (1+\gamma)(m+\ell-1)$. $\boxtimes$

The following key statement is essentially a corollary of Proposition 3.

Theorem 7 *Let $\mathcal{F}_n(I)$ be as in Theorem 2, $I_0 \subset I$ be a compact interval satisfying Property F. Then there is a constant $M_0 \geq 4$ such that for any $\mathbf{r} \in \mathcal{R}_n$ and any $\mathbf{f} \in \mathcal{F}_n(I)$ the set $\mathbf{f}^{-1}(\mathbf{Bad}(\mathbf{r}))$ is M-Cantor rich in I_0 for any $M > \max\{M_0, 16^{1+1/\tau}\}$, where $\tau = \tau(\mathbf{r})$ is defined by* (5).

Proof. Let R_0 and m_0 be as in Proposition 3 and $M_0 = \max\{R_0, 4^{(2n)^4}\}$. Let $M > \max\{M_0, 16^{1+1/\tau}\}$, $R \geq M$ and $m \geq m_0$. Take any $\mathbf{f} \in \mathcal{F}_n(I)$ and

$\mathbf{r} \in \mathcal{R}_n$. Let $(\mathcal{I}_q)_{q\geq 0}$ denote the R-sequence in I_0 that arises from Proposition 3. By (48) and Lemma 1, we have that $\mathcal{K}(\mathcal{I}_q) \subset \mathbf{f}^{-1}(\mathbf{Bad}(\mathbf{r}))$. Thus, by definition, the fact that $\mathbf{f}^{-1}(\mathbf{Bad}(\mathbf{r}))$ is M-Cantor rich in I_0 will follow on showing that $d(\mathcal{I}_q)$ can be made $\leq \varepsilon$ for any $\varepsilon > 0$.

Observe that $(1-\lambda)/2 = \tau/(2+2\tau)$. Then, since $R \geq M > 16^{1+1/\tau}$, we have that $4R^{-\frac{1-\lambda}{2}} < 1$. By conditions (ii) and (iii) of Proposition 3, for $q > 0$

$$\sum_{p=1}^{q-1} \left(\frac{4}{R}\right)^{q-p} \max_{I_p \in \mathcal{I}_p} \#\big(\widehat{\mathcal{I}}_{q,p} \sqcap I_p\big) \ll \sum_{q-p\geq m-3} \left(\frac{4}{R}\right)^{q-p} R^{\frac{1+\lambda}{2}(q-p)-\frac{1-\lambda}{2}m+3} <$$

$$< R^{-\frac{1-\lambda}{2}m+3} \sum_{\ell \geq m-3} \left(4R^{-\frac{1-\lambda}{2}}\right)^{\ell} = R^{-\frac{1-\lambda}{2}m+3} \frac{\left(4R^{-\frac{1-\lambda}{2}}\right)^{m-3}}{1-4R^{-\frac{1-\lambda}{2}}} \to 0 \qquad (66)$$

as $m \to \infty$. Further, since $R \geq M > M_0 \geq 4^{(2n)^n}$, we have that $4R^{-(1-\sigma)} = 4R^{-(2n)^{-4}} < 1$. Once again, by conditions (ii) and (iii) of Proposition 3, for $q \leq m$ we have that $\#\big(\widehat{\mathcal{I}}_{q,0} \sqcap I_0\big) = 0$, while for $q > m$

$$\left(\frac{4}{R}\right)^{q} \#\big(\widehat{\mathcal{I}}_{q,0} \sqcap I_0\big) \ll \left(\frac{4}{R}\right)^{q} R^{\sigma q} = \left(4R^{-(1-\sigma)}\right)^{q} < \left(4R^{-(2n)^{-4}}\right)^{m} \to 0 \quad (67)$$

as $m \to \infty$. By (43), combining (66) and (67) gives $d_q(\mathcal{I}_q) \leq \varepsilon$ for all $q > 0$ provided that m is sufficiently large. This completes the proof. ⊠

Proof of Theorem 2. Let I_0 and M_0 be the same as in Theorem 7 and $M = \max\left\{M_0, 16^{1+1/\tau_0}\right\}+1$, where $\tau_0 = \inf_{k\in\mathbb{N}} \tau(\mathbf{r}_k)$. By (6), $\tau_0 > 0$ and so $M < \infty$. By Theorem 7, each set $\mathbf{f}^{-1}(\mathbf{Bad}(\mathbf{r}_k))$ is M-Cantor rich in I_0. By Theorem 6, so is $S = \bigcap_{\mathbf{f}\in\mathcal{F}_n(I)} \bigcap_{k\in\mathbb{N}} \mathbf{f}^{-1}(\mathbf{Bad}(\mathbf{r}_k))$. By Theorem 5, $\dim S = 1$. By design, (10) holds. The proof is thus complete. ⊠

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